Mathematics. Olympiad trainers league

SOLUTIONS, COMMENTS AND MARKING SCHEMES

I. *Mathematical Set*

1. Trigonometry. Calculate:

$$\tan\frac{\pi}{92}\cdot\tan\frac{2\pi}{92}+\tan\frac{2\pi}{92}\cdot\tan\frac{3\pi}{92}+\ldots+\tan\frac{k\pi}{92}\cdot\tan\frac{(k+1)\pi}{92}+\ldots+\tan\frac{2022\pi}{92}\cdot\tan\frac{2023\pi}{92}$$

Answer: -2024.

Solution. Using the formula

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta},$$

we get

$$\tan\alpha\cdot\tan\beta=\frac{\tan\alpha-\tan\beta}{\tan(\alpha-\beta)}-1.$$

Hence

$$\tan\frac{k\pi}{92}\cdot\tan\frac{(k+1)\pi}{92}=\frac{\tan\frac{(k+1)\pi}{92}-\tan\frac{k\pi}{92}}{\tan\frac{\pi}{92}}-1.$$

Therefore

$$\tan\frac{\pi}{92}\cdot\tan\frac{2\pi}{92}+\tan\frac{2\pi}{92}\cdot\tan\frac{3\pi}{92}+\ldots+\tan\frac{k\pi}{92}\cdot\tan\frac{(k+1)\pi}{92}+\ldots+\tan\frac{2022\pi}{92}\cdot\tan\frac{2023\pi}{92}=$$
$$\frac{1}{\tan\frac{\pi}{92}}\left(\tan\frac{2\pi}{92}-\tan\frac{\pi}{92}+\tan\frac{3\pi}{92}-\tan\frac{2\pi}{92}+\ldots+\tan\frac{2023\pi}{92}-\tan\frac{2022\pi}{92}\right)-2022=$$
$$=\frac{1}{\tan\frac{\pi}{92}}\left(\tan\frac{2023\pi}{92}-\tan\frac{\pi}{92}\right)-2022=\frac{1}{\tan\frac{\pi}{92}}\left(\tan\left(22\pi-\frac{\pi}{92}\right)-\tan\frac{\pi}{92}\right)-2022=$$
$$=\frac{1}{\tan\frac{\pi}{92}}\left(-2\tan\frac{\pi}{92}\right)-2022=-2024.$$

Marking scheme

1. Proved that $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$ **(3 points)**

2. The expression is transformed to

$$\frac{1}{\tan\frac{\pi}{92}}\left(\tan\frac{2\pi}{92}-\tan\frac{\pi}{92}+\tan\frac{3\pi}{92}-\tan\frac{2\pi}{92}+\ldots+\tan\frac{2023\pi}{92}-\tan\frac{2022\pi}{92}\right)-2022$$

(**3 points**)

3. The expression is equal to

$$\frac{1}{\tan \frac{\pi}{92}} \left(\tan \frac{2023\pi}{92} - \tan \frac{\pi}{92} \right) - 2022$$

(**2 points**)

3. Correct answer is obtained (**2 points**)

Penalties: minus 1-2 points for each mistake.

2. **Numbers.** 172 different positive integers are written on the board, and the largest of them does not exceed 550. It is known that among these numbers there are no two whose difference is 4, 5 or 9. Prove that the number 275 is written on the board.Prove that the number 275 is written on the board.

Proof.

Lemma. *Among any* 13 *consecutive natural numbers, it is possible to select no more than four such that no two of them differ by* 4, 5, *or* 9.

Proof of the lemma. Divide the numbers from a to $a+12$ into 9 groups (of one or two numbers) and write the groups in a circular order as follows: $\{a+4\}$, $\{a, a+9\}$, $\{a+5\}$, $\{a+1, a+10\}$, $\{a+6\}$, $\{a+2, a+11\}$, $\{a+7\}$, $\{a+3, a+12\}$,
$\{a+8\}$.

If five or more numbers are selected, then some two of them will be in the same group or in neighboring groups. However, from two neighboring groups, no more than one number can be selected. The lemma is proven.

Let us mark the four middle numbers 274, 275, 276, 277 and divide all other numbers from 1 to 550 into $\frac{550-4}{13} = 42$ groups of 13 consecutive numbers (this is possible because 273 is divisible by 13). From the lemma, it follows that in groups of 13 numbers, no more than $42 \cdot 4 = 168$ numbers can be selected in the required manner. Therefore, the marked 4 numbers are selected.

Marking scheme

1. Proof of the lemma (**4 points**)

2. Splitting into 42 groups of 13 consecutive numbers. (**2 points**)

3. Proof that that in groups of 13 numbers, no more than $42 \cdot 4 = 168$ numbers can be selected in the required manner (**2 points**)

4. Complete the proof (**2 points**)

3. Geometry. Triangle ABC is inscribed in a circle ω. Let A_0 and C_0 be the midpoints of the arcs BC and AB on ω, not containing the opposite vertex, respectively. The circle ω_1 centered at A_0 is tangent to BC, and the circle ω_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to ω_1 and ω_2.

Proof. Let B_0 be the midpoint of arc AC. Note that AIA_0, BIB_0 and CIC_0 are each collinear triples of points. We now have that

$$\angle C_0BI = \angle C_0BB_0 = \tfrac{1}{2}\angle B + \tfrac{1}{2}\angle C = 90^\circ - \angle A.$$

Furthermore, $\angle BC_0I = \angle BC_0C = \angle A$. Thus C_0BI is isosceles with $C_0B = C_0I$. Similarly, we have $A_0B = A_0I$. Thus I is the reflection of B over A_0C_0.

Let D and E be the points at which the tangents from B to ω_1 and ω_2 other than BA and BC touch these circles. We have that

$$\angle DBE = 2\angle C_0BA + \angle B + 2\angle CBA_0 = \angle C + \angle B + \angle A = 180^\circ.$$

Thus a common tangent to ω_1 and ω_2 passes through A. Reflecting about A_0C_0 gives that I also lies on a common tangent to ω_1 and ω_2. ■

Marking scheme

1. Proved that $\angle C_0BI = 90^\circ - \angle A$ oe **(2 points)**
2. Proved that $C_0B = C_0I$ and / or $A_0B = A_0I$ **(2 points)**
3. Proved that $\angle DBE = 180^\circ$ **(2 points)**
4. Proved that a common tangent to ω_1 and ω_2 passes through A **(2 points)**
5. Proof is completed **(2 points)**

II. *Methodical Set*

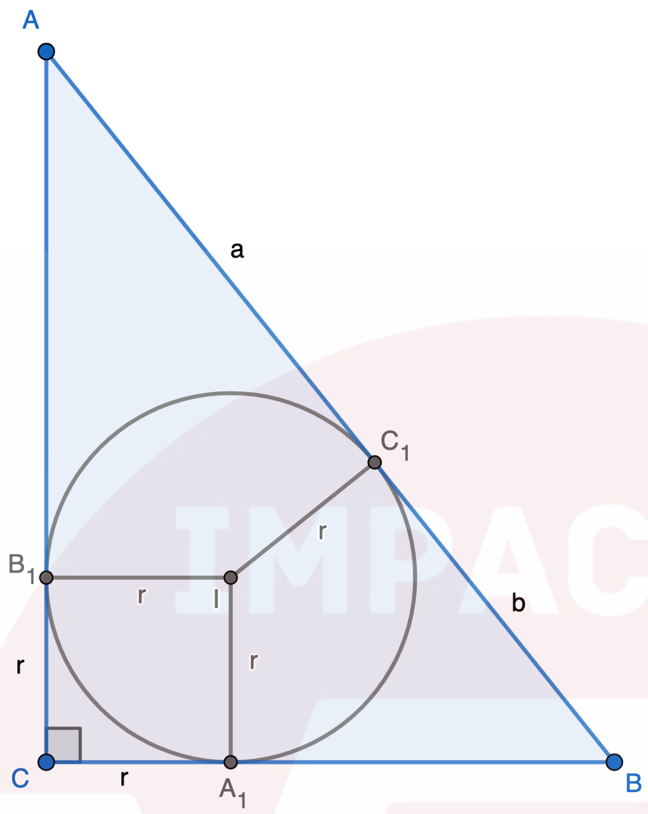
4. Right Triangle. Give four different solutions to the problem below:

Problem

A circle is inscribed in a right triangle. The circle divides the hypotenuse into two segments of the lengths a and b. Find the area of the triangle.

Answer: ab.

Solution. Let $\triangle ABC$ be a right triangle, $\angle C = 90^\circ$, I be the incenter of the incircle. The circle touches sides BC, AC, AB in points A_1, B_1, C_1 respectively. Denote $AC_1 = AB_1 = a$, $BC_1 = BA_1$, and let $CA_1 = CB_1 = r$.



Proof 1 (using Pythagoras' Theorem).

By Pythagoras' Theorem in $\triangle ABC$,

$$AB^2 = AC^2 + BC^2,$$

so $(a+b)^2 = (a+r)^2 + (b+r)^2$.

Solving the equation, we get that

$$r = \frac{1}{2}\left(-(a+b) + \sqrt{a^2+6ab+b^2}\right).$$

Therefore the area of the triangle ABC is

$$\frac{1}{2}AC \times BC = \frac{1}{2}(a+r)(b+r) =$$

$$\frac{1}{8}\left(a-b+\sqrt{a^2+6ab+b^2}\right)\left(-(a-b)+\sqrt{a^2+6ab+b^2}\right) =$$

$$= \frac{1}{8}\left(a^2+6ab+b^2-a^2+2ab-b^2\right) = ab.$$

Proof 2 (using the formula $S = pr$, where p is the semi-perimeter).

$$S_{\triangle ABC} = pr = (AB + AC + BC)r =$$

$$\frac{a+b+a+r+b+r}{2} \cdot r =$$

$$\frac{2a+2b+2r}{2} \cdot r = (a+b+r)r.$$

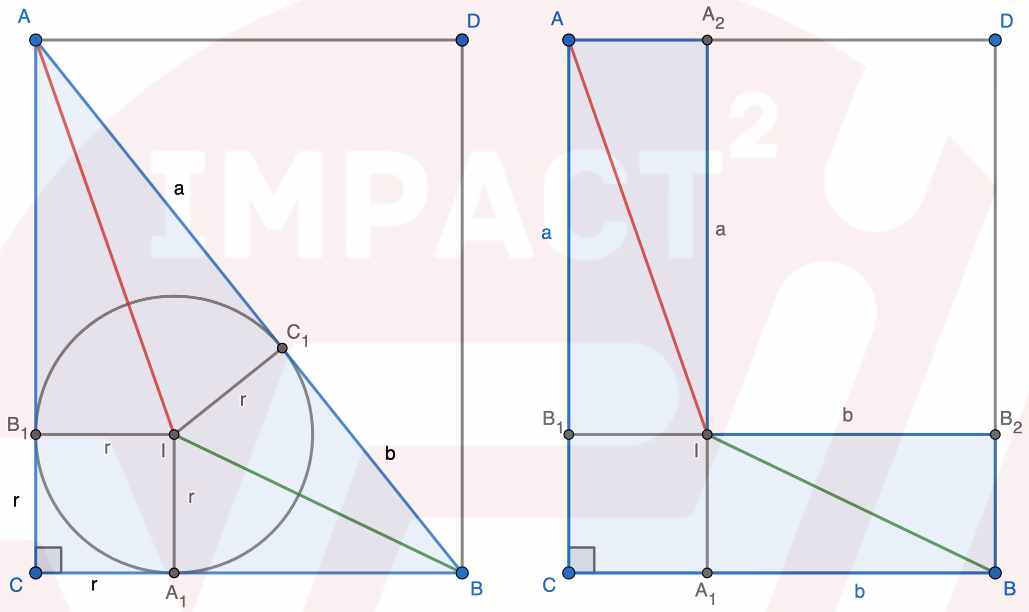
On the other hand,

$$2S_{\triangle ABC} = AC \times BC = \frac{1}{2}(a+r)(b+r) = ab + (a+b+r)r.$$

Hence

$$2S_{\triangle ABC} = ab + S_{\triangle ABC} \iff S_{\triangle ABC} = ab.$$

Proof 3 (rearranging).



$$S_{\triangle ABC} = \tfrac{1}{2}S_{ACBD} = S_{AA_2IB_2BC} = S_{A_2DB_2I} = ab.$$

Proof 4 (using complex numbers). Let us denote $\rho_1 = AI$ and $\rho_2 = BI$. Then

$$r = \rho_1 \cos\varphi_1 = \rho_2 \cos\varphi_2,$$
$$a = \rho_1 \sin\varphi_1, b = \rho_2 \sin\varphi_2,$$

where $\varphi_1 = \angle AIC_1$, $\varphi_2 = \angle BIC_1$.

We have

$$\varphi_1 + \varphi_2 = \angle AIB = 90^\circ + \frac{\angle C}{2} = 90^\circ + \frac{90^\circ}{2} = 135^\circ = \frac{3\pi}{4}.$$

Considering complex numbers

$$z_1 = r + ai = \rho_1 \cos\varphi_1 + \rho_1 i \sin\varphi_1 = \rho_1(\cos\varphi_1 + i\sin\varphi_1) = \rho_1 e^{\varphi_1},$$
$$z_2 = r + bi = \rho_2 \cos\varphi_2 + \rho_2 i \sin\varphi_2 = \rho_2(\cos\varphi_2 + i\sin\varphi_2) = \rho_2 e^{\varphi_2}.$$

Hence

$$z_0 = z_1 z_2 = \rho_1\rho_2 e^{\varphi_1+\varphi_2} = e^{\frac{3\pi}{4}},$$

Since
$$\arg z_0 = \frac{3\pi}{4}$$
, $\mathrm{Re} z_0 + \mathrm{Im} z_0 = 0$.

Also, we have
$$z_0 = (r+ai)(r+bi) = (r^2 - ab) + ir(a+b)) \iff \mathrm{Re} z_0 = r^2 - ab, \mathrm{Im} z_0 = r(a+b).$$
Thus
$$(r^2 - ab) + r(a+b) = 0 \iff r^2 + r(a+b) = ab.$$
Therefore
$$S_{ABC} = \frac{1}{2} AC \cdot BC = \frac{1}{2}(r+a)(r+b) = \frac{1}{2}(r^2 + r(a+b) + ab) = \frac{1}{2}(ab+ab) = \frac{1}{2} \cdot (2ab) = ab.$$

Marking scheme

1. 4 proofs or more (**10 points**)
2. 3 proofs (**8 points**)
3. 2 proofs (**5 points**)
4. 1 proof (**2 points**)
5. 0 proofs (**0 points**)

Penalties: minus 1-2 points for each mistake.

5. **Coordination.** The following problem was proposed at Mathematical Olympiad:

Problem

Let $a > 1$ be a positive integer. Let M be the set of positive integers m such that all of the prime divisors of $a^m - 1$ are less than 10^{2023}. Prove that M is finite.

Below you can find a **«partial solution»** to the problem given by a student. Using the result obtained by the student, complete the solution of the problem. Specify your proposals at the coordination, with justification, for a possible assessment of the student's «partial solution» (full correct solution, as normal, is worth 7 **points**).

Student's «partial solution»

Let $P = \{p_1, p_2, \ldots, p_n\}$ be the primes less than 10^{2023} which do not dived a, and assume that there are infinitely many m such that all prime divisors of $a^m - 1$ are in P.

Let e_i be the order of a modulo p_i, so that $p_i | a^m - 1$ if and only if $e_i | m$. Let $f_i = v_p\left(a^{2e_i} - 1\right)$, and then by LTE lemma we have
$$v_{p_i}\left(a^m - 1\right) \leqslant v_{p_i}\left(a^{2m} - 1\right) = v_{p_i}\left(a^{2e_i - 1} - 1\right) + v_{p_i}\left(\frac{m}{e_i}\right) \leqslant f_i + v_{p_i}(m).$$

The end of the solution to the problem.

Thus

$$a^{n_j} - 1 = \prod_{i=1}^{r} p_i^{v_p(a^{n_j}-1)} \leqslant \prod_{i=1}^{r} p_i^{f_i + v_p(n_j)} = n_j \prod_{i=1}^{r} p_i^{f_i}$$

where $\prod_{i=1}^{r} p_i^{f_i} = C$ is a constant. Hence, for all j, $a^{n_j} - 1 \leqslant Cn_j$, but this is false for sufficiently large n. Thus, M is finite.

Comments.

Possible assessment for the student's «partial solution»: **4 points out of 7**.

The student made a typo in the last inequality, it has to be written $v_{p_i}\left(a^{2e_i} - 1\right)$ instead of $v_{p_i}\left(a^{2e_i-1} - 1\right)$.

Justification. Here is a possible **Marking Scheme** for the problem:

- Proved that $v_{p_i}(a^m - 1) \leqslant v_{p_i}\left(a^{2e_i-1} - 1\right) + v_{p_i}\left(\frac{m}{e_i}\right)$ **(3 points)**
- Hence proved that $v_{p_i}(a^m - 1) \leqslant f_i + v_{p_i}(m)$ **(1 point)**
- Proved that for all j, $a^{n_j} - 1 \leqslant Cn_j$, where C is constant **(2 points)**
- Hence, it is shown that M is infinite **(1 point)**

Marking scheme

1. Proved that for all j, $a^{n_j} - 1 \leqslant Cn_j$, where C is constant **(3 points)**
2. Hence, it is shown that M is infinite **(2 point)**
3. Student's «partial solution» was been assessed **(2 points)**
4. Justification of the assessment (for example, a possible marking scheme was been demonstrated) **(3 points)**
5. Noticed that the student made a typo **(1 point)**

Maximum mark for the problem is 10.

6. **Searching for mistakes.** The following problem was proposed at Mathematical Olympiad:

Problem

Psychologists decided to create a new personality type classifier. 100 traits were selected to describe personality types. Each trait can be present or absent. The classifier is considered good if the number of simultaneously present or absent traits in any two personality types is less than 50. Determine the maximum number of personality types that can be contained in a good classifier.

Below you can find **«solution»** to the problem given by a student. The «solution» may contain mathematical mistakes. If the student's «solution» is incorrect, then indicate all the mistakes and give the correct solution.

Student's «solution»

Choose two personality types uniformly at random from a group of n types. Let X be the number of traits in which these two types differ. X represents the total number of such differences between the two chosen personality types.

Let X_i be a binary random variable that is equal to 1 if the two personality types in the chosen pair have opposite i-th traits, and 0 otherwise. Then

$$X = \sum_{i=1}^{100} X_i.$$

Thus, the expectation of X can be found by the formula:

$$\mathbb{E}[X] = \sum_{i=1}^{100} \mathbb{E}[X_i].$$

For $i \in \{1, 2, \dots, 100\}$, let n_i be the number of personality types with ith trait present. Then the number of pairs of personality types that have opposite ith trait is $n_i(n-n_i)$. Therefore,

$$\mathbb{E}[X_i] = \frac{n_i(n-n_i)}{\binom{n}{2}} = \frac{2n_i(n-n_i)}{n(n-1)}.$$

By AM-GM,

$$\sqrt{n_i(n-n_i)} \leqslant \frac{n_i + (n-n_i)}{2} = \frac{n}{2}.$$

So,

$$\mathbb{E}[X_i] = \frac{2n_i(n-n_i)}{n(n-1)} \leqslant \frac{2n^2}{4n(n-1)} = \frac{n}{2(n-1)}.$$

Hence,

$$\mathbb{E}[X] = \sum_{i=1}^{100} \mathbb{E}[X_i] \leqslant \sum_{i=1}^{100} \frac{n}{2(n-1)} = \frac{50n}{2(n-1)}.$$

When $n = 51$, $\mathbb{E}[X] = 51$. When $n = 52$, we get $\mathbb{E}[X] < 51$. Since X is an integer, when there must exist a pair of personality types such that the number of opposite traits this pair has is $\leqslant 50$. In other words, there exists a pair of personality types that have 50 or more common traits. Therefore, a good classifier cannot describe 52 personality types.

«Answer:» 51.

Mistakes and a correct solution

1. The student made a mistake in this inequality:

$$\mathbb{E}[X] = \sum_{i=1}^{100} \mathbb{E}[X_i] \leqslant \sum_{i=1}^{100} \frac{n}{2(n-1)} = \cancel{\frac{50n}{2(n-1)}} \to \frac{50n}{n-1}.$$

2. The student didn't demonstrated an example when good classifier has exactly 51 personality types.

3. When $n = 51$, $\mathbb{E}[X] = 51$ if and only if

$$n_i = n - n_i \iff n_i = \frac{n}{2} = \frac{51}{2}.$$

4. n_i is an integer, the equality cannot hold when $n = 51$. That is, when $n = 51$, $\mathbb{E}[X] < 51$. Since X is an integer, when there must exist a pair of personality types such that the number of opposite traits this pair has is < 50. In other words, there exists a pair of personality types that have 50 or more common traits. Therefore, a good classifier cannot describe 51 personality types. We can construct an example (by induction) of a good classifier for $n = 50$.

5. So, the answer is incorrect. The answer is 50.

Marking scheme

1. Specified that the answer is incorrect (**1 point**)
2. Noticed that the student made a typo in the inequality (**1 point**)
3. Proved that, when $n = 51$, $\mathbb{E}[X] < 51$ (**4 points**)
4. Proved that $n = 51$ does not work (**1 point**)
5. Noticed that the student didn't demomstrated an example for $n = 51$ (**1 point**)
6. Given the idea how to construct an example for $n = 50$ (**2 points**)

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1. Trigonometry. Calculate:

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Answer: -2024.

Solution. Using the formula

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta},$$

we get

$$\tan\alpha\cdot\tan\beta=\frac{\tan\alpha-\tan\beta}{\tan(\alpha-\beta)}-1.$$

Hence

$$\tan\frac{k\pi}{92}\cdot\tan\frac{(k+1)\pi}{92}=\frac{\tan\frac{(k+1)\pi}{92}-\tan\frac{k\pi}{92}}{\tan\frac{\pi}{92}}-1.$$

Therefore

$$\tan\frac{\pi}{92}\cdot\tan\frac{2\pi}{92}+\tan\frac{2\pi}{92}\cdot\tan\frac{3\pi}{92}+\ldots+\tan\frac{k\pi}{92}\cdot\tan\frac{(k+1)\pi}{92}+\ldots+\tan\frac{2022\pi}{92}\cdot\tan\frac{2023\pi}{92}=$$
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$$=\frac{1}{\tan\frac{\pi}{92}}\left(-2\tan\frac{\pi}{92}\right)-2022=-2024.$$

Marking scheme

1. Proved that $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$ **(3 points)**

2. The expression is transformed to

$$\frac{1}{\tan\frac{\pi}{92}}\left(\tan\frac{2\pi}{92}-\tan\frac{\pi}{92}+\tan\frac{3\pi}{92}-\tan\frac{2\pi}{92}+\ldots+\tan\frac{2023\pi}{92}-\tan\frac{2022\pi}{92}\right)-2022$$

(**3 points**)

3. The expression is equal to

$$\frac{1}{\tan\frac{\pi}{92}}\left(\tan\frac{2023\pi}{92}-\tan\frac{\pi}{92}\right)-2022$$

(**2 points**)

3. Correct answer is obtained (**2 points**)

Penalties: minus 1-2 points for each mistake.

2. Numbers. 172 different positive integers are written on the board, and the largest of them does not exceed 550. It is known that among these numbers there are no two whose difference is 4, 5 or 9. Prove that the number 275 is written on the board.Prove that the number 275 is written on the board.

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Proof. Let B_0 be the midpoint of arc AC. Note that AIA_0, BIB_0 and CIC_0 are each collinear triples of points. We now have that

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Thus a common tangent to ω_1 and ω_2 passes through A. Reflecting about A_0C_0 gives that I also lies on a common tangent to ω_1 and ω_2. ∎

Marking scheme

1. Proved that $\angle C_0BI = 90^\circ - \angle A$ oe (**2 points**)
2. Proved that $C_0B = C_0I$ and / or $A_0B = A_0I$ (**2 points**)
3. Proved that $\angle DBE = 180^\circ$ (**2 points**)
4. Proved that a common tangent to ω_1 and ω_2 passes through A (**2 points**)
5. Proof is completed (**2 points**)

II. *Methodical Set*

4. **Right Triangle.** Give four different solutions to the problem below:

Problem

A circle is inscribed in a right triangle. The circle divides the hypotenuse into two segments of the lengths a and b. Find the area of the triangle.

Proof 1.

Proof 2.

Proof 3.

Proof 4.

Marking scheme

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2. 3 proofs	(**8 points**)
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$$v_{p_i}\left(a^m - 1\right) \leqslant v_{p_i}\left(a^{2m} - 1\right) = v_{p_i}\left(a^{2e_i - 1} - 1\right) + v_{p_i}\left(\frac{m}{e_i}\right) \leqslant f_i + v_{p_i}(m).$$

The end of the solution to the problem.

Marking scheme

6. Searching for mistakes. The following problem was proposed at Mathematical Olympiad:

Problem

Psychologists decided to create a new personality type classifier. 100 traits were selected to describe personality types. Each trait can be present or absent. The classifier is considered good if the number of simultaneously present or absent traits in any two personality types is less than 50. Determine the maximum number of personality types that can be contained in a good classifier.

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Choose two personality types uniformly at random from a group of n types. Let X be the number of traits in which these two types differ. X represents the total number of such differences between the two chosen personality types.

Let X_i be a binary random variable that is equal to 1 if the two personality types in the chosen pair have opposite i-th traits, and 0 otherwise. Then

$$X = \sum_{i=1}^{100} X_i.$$

Thus, the expectation of X can be found by the formula:

$$\mathbb{E}[X] = \sum_{i=1}^{100} \mathbb{E}[X_i].$$

For $i \in \{1, 2, \ldots, 100\}$, let n_i be the number of personality types with ith trait present. Then the number of pairs of personality types that have opposite ith trait is $n_i(n-n_i)$. Therefore,

$$\mathbb{E}[X_i] = \frac{n_i(n-n_i)}{\binom{n}{2}} = \frac{2n_i(n-n_i)}{n(n-1)}.$$

By AM-GM,

$$\sqrt{n_i(n-n_i)} \leqslant \frac{n_i + (n-n_i)}{2} = \frac{n}{2}.$$

So,

$$\mathbb{E}[X_i] = \frac{2n_i(n-n_i)}{n(n-1)} \leqslant \frac{2n^2}{4n(n-1)} = \frac{n}{2(n-1)}.$$

Hence,

$$\mathbb{E}[X] = \sum_{i=1}^{100} \mathbb{E}[X_i] \leqslant \sum_{i=1}^{100} \frac{n}{2(n-1)} = \frac{50n}{2(n-1)}.$$

When $n = 51$, $\mathbb{E}[X] = 51$. When $n = 52$, we get $\mathbb{E}[X] < 51$. Since X is an integer, when there must exist a pair of personality types such that the number of opposite traits this pair has is $\leqslant 50$. In other words, there exists a pair of personality types that have 50 or more common traits. Therefore, a good classifier cannot describe 52 personality types.

«Answer:» 51.

Mistakes and a correct solution

Marking scheme

Problem 2

На доске записаны 172 различных натуральных числа, причем самое больше не превосходит 550. Известно, что среди этих чисел не существует двух, разность которых равна $4,\ 5$ или 9. Докажите, что число 275 записано на доске.

Лемма. Среди любых 13 подряд идущих натуральных чисел можно выбрать не более четырех так, что никакие два из них не различаются на $4,\ 5$ или 9.
Доказательство. Разобьем 13 чисел с a до $a+12$ на 9 групп (из одного или двух чисел) и
запишем группы по кругу в следующем порядке:
$\{a+4\},\ \{a, a+9\},\ \{a+5\},\ \{a+1, a+10\},\ \{a+6\},\ \{a+2,\ a+11\}, \{a+7\}, \{a+3, a+12\},$
$\{a+8\}$

Если выбрано 5 или более чисел, то некоторые два из них окажутся в одной группе или в соседних группах. Однако из двух соседних групп можно выбрать не более одного числа. Лемма доказана.

Отметим 4 средних числа $274,\ 275,\ 276,\ 277,$ а все остальные числа от 1 до 550 разобьем на $\frac{550-4}{13} = 42$ группы по 13 последовательных чисел (это возможно, так как 273 делится на 13). Из леммы следует, что в группах по 13 чисел можно выбрать не более $42 \cdot 4 = 168 - 4$ числа требуемым в условии образом. Значит, отмеченные 4 числа выбраны.